MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

ADVANCED MATRICULATION LEVEL
2024 SECOND SESSION

SUBJECT:	**Pure Mathematics**
PAPER NUMBER:	I
DATE:	29^{th} August 2024
TIME:	09:00 to 12:05

Directions to Candidates

Answer **ALL TEN** questions.
Each question carries 10 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

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1. (a) Express $\dfrac{1}{y(y+1)}$ and $\dfrac{1}{x(x-1)}$ into partial fractions. **[4 marks]**

   (b) Hence, solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y(y+1)}{x(x-1)},$$

   given that $y = 2$ when $x = 3$. Give your final answer in the form $y = f(x)$. **[6 marks]**

2. (a) Let $y = \cos(2e^{-x})$. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$, and show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} + 4ye^{-2x} = 0.$$

   [5 marks]

   (b) A curve has equation $2y^3 - 5xy^2 + 4x^2y + 4x^3 = 4$. Find the equation of the line that is normal to the curve at the point $(2, -1)$. **[5 marks]**

3. The line ℓ_1 is given by the vector equation

$$\mathbf{r} = 7\mathbf{i} + 3\mathbf{j} + 8\mathbf{k} + \lambda(3\mathbf{i} + \mathbf{j} + 5\mathbf{k}),$$

   and the point A has position vector $2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$.

   (a) Find the position vector of the point B on ℓ_1 such that AB is perpendicular to ℓ_1. Find the distance from A to B. **[5 marks]**

   (b) Write down the equation of the line ℓ_2 passing through A and B. Find the coordinates of the point where this line intersects the xy-plane. **[5 marks]**

4. (a) The function f is given by $f(x)=(x+3)(x+1)(x-1)$.
   (i) Sketch the curve of $y = f(x)$, showing clearly where the curve cuts the coordinate axes.
   (ii) On the same set of axes, sketch the curve of $y = f(2x-1)$.

[3, 4 marks]

(b) Find the values of a, b and c given that the matrices P and Q given below are inverses of each other:

$$P = \begin{pmatrix} 1 & -1 & 1 \\ 2 & a & 3 \\ -2 & 1 & 2 \end{pmatrix} \qquad \text{and} \qquad Q = \begin{pmatrix} b & 3 & -1 \\ -10 & 4 & -1 \\ -2 & 1 & c \end{pmatrix}.$$

[3 marks]

5. (a) Use integration by parts to evaluate the integral

$$\int_1^2 x^2 \ln x \, dx.$$

[5 marks]

(b) Use the substitution $u = 1 + \sin x$ to find the integral

$$\int \cos x \sin x (1+\sin x)^3 \, dx.$$

[5 marks]

6. Let α and β be the roots of the quadratic equation

$$x^2 + \sqrt{2}\, x \cos\theta + \sin^2\theta = 0.$$

(a) Show that the quadratic equation with roots $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$ is

$$x^2 \sin^2\theta - 2x\cos 2\theta + \sin^2\theta = 0. \qquad (\star)$$

[5 marks]

(b) Find the values of θ in the range $0 \leq \theta < 360°$ for which the quadratic equation $(\star)$ has repeated roots. (Give your answers correct to two decimal places.) **[5 marks]**

7. (a) Calculate the sum of all the multiples of 3 from 3 to 99 inclusive,

$$3 + 6 + 9 + \cdots + 99.$$

[3 marks]

(b) Consider the arithmetic series

$$4p + 8p + 12p + \cdots + 400$$

where p is a positive integer and a factor of 100.
   (i) Find an expression for the number of terms in this series in terms of p.
   (ii) Show that the sum of this series is $q + \dfrac{100q}{p}$ for some positive integer q. Find q.

[1, 3 marks]

(c) Write the recurring decimal $0.123123123\ldots$ as the sum of a geometric series. Hence write the recurring decimal as a rational number. **[3 marks]**

8. Points with coordinates (x, y) are plotted on the xy-plane using only the integers between 1 and 10 (both inclusive) for x and y.
   (a) How many points can be plotted in total? **[2 marks]**
   (b) What is the probability that a point chosen at random from those plotted has the y-coordinate greater than the x-coordinate? **[3 marks]**
   (c) Write down the equations of the two lines ℓ_1 and ℓ_2 passing through the plotted points whose x-coordinate and y-coordinate differ by exactly 2. **[2 marks]**
   (d) Find the perpendicular distance between the lines ℓ_1 and ℓ_2. **[3 marks]**

9. Consider the function $f(x) = \sqrt{x+2} - 7$ defined for $x \in \mathbb{R}$, $x \geq -2$.
   (a) Determine the inverse function $f^{-1}(x)$, giving its domain and range. **[3, 2, 1 marks]**
   (b) Solve the equation $f^{-1}(x) = f(62)$. **[4 marks]**

10. Let $i = \sqrt{-1}$.
    (a) Given that $z = 1 - 2\sqrt{2}i$ express $\sqrt{z}$ and $\dfrac{1}{\sqrt{z}}$ in the form $p\sqrt{n} + iq$, where p and q are rational numbers, and n is a positive integer. **[5 marks]**
    (b) Show that $z = 2 - 5i$ is a root of the equation
    $$z^3 - 7z^2 + 41z - 87 = 0.$$
    Find the other two roots. **[5 marks]**

AM 27/II.24S

MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

ADVANCED MATRICULATION LEVEL
2024 SECOND SESSION

SUBJECT:	**Pure Mathematics**
PAPER NUMBER:	II
DATE:	30th August 2024
TIME:	09:00 to 12:05

Directions to Candidates

Answer **SEVEN** questions from **TEN**. Each question carries 15 marks.
Graphical calculators are **not** allowed however scientific calculators can be used but all necessary working must be shown.

1. (a) (i) Use the substitution $u = x^4$ to find the integral $\displaystyle\int x^3 e^{x^4}\, dx$.

   (ii) Hence, find the particular solution to the differential equation

$$\frac{dy}{dx} + 4x^3 y = x^3,$$

given that $y = 10$ when $x = 0$. Give your answer in the form $y = f(x)$. **[3, 5 marks]**

(b) Solve the differential equation

$$\frac{d^2y}{dx^2} + 4y = 6x + 1,$$

given that $y = \frac{11}{4}$ and $\dfrac{dy}{dx} = 1$ when $x = 0$.

[7 marks]

2. (Note that angles should be taken in radians throughout this question.)

(a) Show that the equation $\ln(3x^2 + 1) = 5e^{-x} - 2$ has a solution between 0 and 1. Use the Newton-Raphson method to find an approximate value of this solution, taking 0 as a first approximation. Do **TWO** iterations and give your working to four decimal places.

[8 marks]

(b) Evaluate $\displaystyle\int_0^{\pi} \cos\left(\frac{x^2}{4} - x\right) dx$ by Simpson's Rule with an interval width of $h = \frac{\pi}{4}$. Give your answer to four decimal places. **[7 marks]**

3. (a) Given that

$$\mathbf{A} = \begin{pmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{pmatrix}$$

evaluate $\det(\mathbf{A})$ and find $\mathbf{A}^{-1}$. **[1, 4 marks]**

(b) The linear transformation

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \mathbf{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where $\mathbf{M}$ is a 3×3 matrix, maps the points with position vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, to the points $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, respectively.

(i) Write down the matrix $\mathbf{M}$ and solve the equation

$$\mathbf{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}.$$

[1, 4 marks]

(ii) Find the Cartesian equation of the plane whose image under the transformation $\mathbf{M}$ is the plane $x = y$. **[5 marks]**

4. (a) (i) Let $I_n = \displaystyle\int_0^1 (1+x)^n e^x \, \mathrm{d}x$. Show that, for $n \geq 1$,

$$I_n = 2^n e - 1 - nI_{n-1}.$$

(ii) The region bounded by the curve $y = (1+x)^{3/2} e^{x/2}$ and the x-axis between $x = 0$ and $x = 1$ is rotated through 2π radians about the x-axis. Find the volume of the solid that is generated by this rotation. **[2, 5 marks]**

(b) A function y is defined by $y = x^{5/2} + \dfrac{1}{5\sqrt{x}}$.

(i) Evaluate $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, and show that $1 + \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \left(\dfrac{25x^3+1}{10x^{3/2}}\right)^2$.

(ii) The part of the curve of this function y between $x = 1$ and $x = 2$ is rotated by 2π radians about the x-axis. Find the area of the surface of revolution so formed.

[3, 5 marks]

5. Let $i = \sqrt{-1}$.
   (a) Use Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$ to show that if $z = \cos\theta + i\sin\theta$ then

$$2\cos n\theta = z^n + \frac{1}{z^n} \qquad \text{and} \qquad 2i\sin n\theta = z^n - \frac{1}{z^n},$$

   for every positive integer n. Hence show that

$$\cos^4\theta\sin^2\theta = \frac{1}{32}(2 + \cos 2\theta - 2\cos 4\theta - \cos 6\theta).$$

[2, 6 marks]

   (b) Find the points of intersection of the loci given by the equations:

$$|z - i| = 2|z + 1| \quad \text{and} \quad |z + 1| = |z + i|.$$

[7 marks]

6. (a) In a group of 30 students, 15 are studying French, 8 are studying Italian and 6 are studying Spanish. There are 3 students studying all the three languages. Show that 7 or more students are not studying any of the three languages. **[6 marks]**
   (b) There are 3 pairs of shoes in a box and each pair is of a different colour. The right shoe is first given at random to 3 children and then the left shoe is given at random to the same 3 children.
      (i) Find the probability that each child gets a matching pair of shoes.
      (ii) It is known that each child wears a different size and that there is a pair of shoes of the right size for each one of them. The children try the shoes that they are given to find out whether they are of the right size for them. What is the probability that each child ends up with a matching pair of shoes and of the right size?

[5, 4 marks]

7. (a) On the same diagram, sketch the curves
      (i) $\mathscr{C}_1$ given by the polar equation $r = 2 + 2\sin\theta$ for $-\pi \le \theta \le \pi$, and
      (ii) $\mathscr{C}_2$ given by the polar equation $r = 2\sin\theta$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

   Shade the area A bounded between the two curves and determine the value of A.

[2, 2, 4 marks]

   (b) Let $f(x) = x(1 + e^x)$. Find the Maclaurin's series of $f(x)$, giving the first four non-zero terms and the term in x^n for $n \ge 2$. **[7 marks]**

8. (a) Show by the method of differences that

$$\sum_{r=1}^{n} \frac{2}{(r+2)(r+4)} = \frac{n(an+b)}{12(n+c)(n+d)},$$

   where the positive integers a, b, c, d are to be determined. **[7 marks]**
   (b) Given that $a_n = \sqrt{2 + a_{n-1}}$ for all integer $n \ge 1$, and $a_0 = 1$, use mathematical induction to prove that for every integer $n \ge 1$,

$$\sqrt{2} < a_n < 2.$$

[8 marks]

9. The function f is given by $f(x) = \dfrac{4x-1}{2x^2+4x-1}$.
   (a) Find where the curve $y = f(x)$ cuts the coordinate axes. **[2 marks]**
   (b) Find the stationary points of the curve $y = f(x)$ and determine their nature. **[6 marks]**
   (c) Determine the equations of any asymptotes of the curve $y = f(x)$. **[3 marks]**
   (d) Sketch the curve of $y = f(x)$. **[4 marks]**

10. The points A and B have position vectors $3\mathbf{i}+\mathbf{j}-2\mathbf{k}$ and $5\mathbf{i}-4\mathbf{j}-3\mathbf{k}$, respectively.
   (a) Find the equation of the plane Π_1 that contains A and B and the line ℓ_1 that passes through B with direction vector $\mathbf{i}+5\mathbf{j}-8\mathbf{k}$. **[5 marks]**
   (b) Find the position vector of the point C that lies on ℓ_1 such that the angle $\angle CAB$ is a right angle. **[5 marks]**
   (c) Find the lengths of the line segments AB, BC and CA (leaving them in surd form) and verify Pythagoras' Theorem for the triangle ABC. **[5 marks]**