MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD
UNIVERSITY OF MALTA, MSIDA

MATRICULATION EXAMINATION
INTERMEDIATE LEVEL

MAY 2015

SUBJECT:	PURE MATHEMATICS
DATE:	22nd May 2015
TIME:	4.00 p.m. to 7.00 p.m.

Directions to candidates

Attempt all questions. There are 10 questions in all.

The marks carried by each question are shown at the end of the question.

The total number of marks for all the questions in the paper is 100.

Graphical calculators are *not* allowed.

Scientific calculators can be used, but all necessary working must be shown.

A booklet with mathematical formulae is provided.

1. (a) Find the value of a if

$$\log_a 45 + 4\log_a 2 - \frac{1}{2}\log_a 81 - \log_a 10 = \frac{3}{2}.$$

(b) Using surds, express the fraction

$$\frac{\sqrt{2}-1}{2\sqrt{2}+3} + \frac{3}{2\sqrt{2}-3}$$

in the form $(a + b\sqrt{2})$ where a and b are negative integers.

[3; 3 marks]

2. (a) The tenth term of an arithmetic progression is 125, and the sum of the first ten terms is 260.

   (i) Show that the first term in the progression is -73, and find the common difference.

   (ii) Find the smallest number, n, if the sum of the first n terms is larger than 1000.

   (b) As a reward for the discovery of the game of chess, its inventor was asked to choose his own prize. He asked for 1 grain of rice to be placed on the first square of the board, 2 grains on the second square, 4 grains on the third square, and so on in geometric progression until all 64 squares had been covered.

   (i) Find the total number of grains of rice he would have received, giving your answer to four significant figures.

   (ii) How many grains of rice would there be on the black squares, assuming the first square in part (i) was black.

[3, 4; 3, 2 marks]

3. At a certain latitude in the northern hemisphere, the number d of hours of daylight in each day of the year is taken to be

$$d = A + B \cos kt,$$

where A, B and k are positive constants, and t is the time in days after the longest day (that is, the day having the maximum number of hours of daylight).

   (i) Assuming that the number of hours of daylight follows an annual cycle of 365 days, and using the periodic properties of the cosine function, verify that

$$k = \frac{2\pi}{365} \text{ radians/day.}$$

   (ii) Given that the shortest and longest days have 6 and 18 hours of daylight respectively, state the values of A and B.

   (iii) Working in radians or otherwise, find, in hours and minutes, the amount of daylight, on the 80th day after the longest day.

   (iv) A town on this latitude holds a fair twice a year on those days having exactly 15 hours of daylight. Find, in relation to the longest day, which two days these are.

[2, 3, 2, 3 marks]

4. The quadratic function $p(x)$ is given by $p(x) = 2x^2 - 3x + 1$.

   (i) Find the equation of the tangent of the graph of $y = p(x)$ at $x = 1$.

   (ii) Using partial fractions, find the constants A and B in the expression
   $$\frac{1}{p(x)} = \frac{A}{2x-1} + \frac{B}{x-1}.$$

   (iii) Using the result obtained in (ii), find the integral $\int \frac{1}{p(x)} dx$.

   (iv) If α and β are the roots of $p(x)$, find the quadratic function, $q(x)$, whose roots are $2\alpha + 1$ and $2\beta + 1$.

[4, 2, 3, 3 marks]

5. AB is a diameter of a circle of radius 5 cm. A chord of this circle, CD, of length 8 cm, intersects AB at the point E, where AE = 1 cm.

   (i) Show that triangles AEC and DEB are similar.

   (ii) Using similarity of triangles, find the lengths of CE and ED.

   (iii) Find the area of the minor segment bound by the circle and the chord CD.

[3, 3, 4 marks]

6. (a) A function $f(x)$ is defined by $f(x) = (1+ax)^n$, where a and n are positive integers. Its binomial expansion is given by $f(x) = 1 + 36x + 576x^2 + \ldots$. Find the values of a and n.

   (b) A fair die has six faces, which are numbered respectively from one to six. Find the probability of obtaining a total score of 5 in three tosses of this die.

   (c) A masters' class consists of 10 English, 12 French and 4 Italian students. A committee of 4 English, 3 French and 2 Italians is to be selected from this class. In how many ways can this be done?

[4; 4; 4 marks]

7. (a) Differentiate the following functions with respect to x:
   $$f(x) = \frac{\cos x}{1+x^2}, \qquad g(x) = e^{2x}(1+x)^3.$$

   (b) An open cylindrical wastepaper bin, of radius r cm and capacity V cm^3, is to have a surface area of 5000 cm^2.

      (i) Show that $V = \frac{1}{2}r(5000 - \pi r^2)$.

      (ii) Calculate the maximum possible capacity of the bin.

[2, 2; 2, 3 marks]

8. (a) A cup of hot coffee left in a draught cools at a rate given by the differential equation

$$\frac{d\theta}{dt} = -4(\theta - 15),$$

where $\theta(t)$ is the temperature in °C, and t is the time in hours. It can be assumed that $\theta = 60$°C when $t = 0$.

   (i) Find $\theta(t)$ using separation of variables.

   (ii) How long does it take for the coffee to cool down to 20°C?

   (b) Using integration, find the area enclosed by the curves $y = x^2$ and $y = 8 - x^2$. Draw a sketch of this area.

[4, 2; 6 marks]

9. (a) A 2×2 matrix $\mathbf{A}$ represents a reflection in the line $y = x$, whilst a 2×2 matrix $\mathbf{B}$ represents a clockwise rotation of $\pi/2$ about the origin.

   (i) Write down the matrices **A** and **B**.

   (ii) Find the matrix **C** defined as $\mathbf{C} = \mathbf{BA} - \mathbf{AB}$.

   (iii) Interpret the matrix **C** geometrically.

   (b) Using the method of the matrix inverse, solve the system of linear equations:

$$\begin{pmatrix} 2 & -1 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \end{pmatrix}.$$

[2, 3, 2; 4 marks]

10. The matrices **A**, **B**, **C**, **x** are given by:

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & -1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 3 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix},$$

where a and b are constants.

If $\mathbf{Bx} = \mathbf{Cx}$ and $\mathbf{x}^t\mathbf{Ax} + \mathbf{Bx} + \frac{3}{8} = 0$, find a and b.

Hint: $\mathbf{x}^t$ denotes the transpose of $\mathbf{x}$.

[6 marks]