MATRICULATION AND SECONDARY EDUCATION CERTIFICATE EXAMINATIONS BOARD

INTERMEDIATE MATRICULATION LEVEL
2024 FIRST SESSION

SUBJECT:	**Pure Mathematics**
DATE:	4th May 2024
TIME:	9:00 a.m. to 12:05 p.m.

Directions to Candidates

Answer **ALL** questions. There are **10** questions in all.
The total number of marks for all the questions in the paper is 100.
Graphical calculators are **not** allowed.
Scientific calculators can be used, but all necessary working must be shown.
A booklet with mathematical formulae is provided.

1. Let $f(x) = x^3 - 5x^2 + 8x - 4$.
   (a) Use the Remainder Theorem to show that $(x - 1)$ is a factor of $f(x)$ and factorise $f(x)$ completely.

   [4 marks]

   (b) Sketch the graph of $y = f(x)$. Deduce that $f(x) \geq 0$ when $x \geq 1$, and $f(x) < 0$ when $x < 1$.

   [2 marks]

   (c) Express into partial fractions $\dfrac{3x^2 - 1}{f(x)}$ and show that

$$\int_3^4 \frac{3x^2-1}{f(x)}\,\mathrm{d}x = \ln(9/2) + \frac{11}{2}.$$

   [3, 3 marks]

   [**Total: 12 marks**]

2. (a) The roots of the equation $x^2 - px + 8 = 0$ are α and $\alpha + 2$. Find the **two** possible values of p.

   [3 marks]

   (b) A function f maps a real number x into the real number y given by

$$y = \frac{\sqrt{2x-1}}{(x+1)(5x-2)}.$$

   Find the domain for which the function f is defined.

   [3 marks]

   [**Total: 6 marks**]

3. (a) Rationalise the denominator of $\dfrac{4-2\sqrt{7}+2\sqrt{3}-\sqrt{21}}{2-\sqrt{7}}$.

[3 marks]

(b) A hollow cone of base radius a and height $5a$ is held vertex downwards. The cone is initially empty and liquid is poured into it at a rate of $5\pi\,\text{cm}^3/\text{s}$.

(i) Show that 243 seconds after the start of the pouring, the depth of the liquid is 45 cm.

(ii) Find the rate at which the depth of the liquid in the vessel is increasing at this time.

[3, 3 marks]

[**Total: 9 marks**]

4. A geometric progression has first term a and common ratio r. The sum of the first two terms is 12, and the third term is 6 less than the first.

(a) Show that

$$\frac{1}{a} = \frac{1+r}{12} = \frac{1-r^2}{6},$$

and find the values of a and r.

[2, 5 marks]

(b) Why does the sum to infinity of this geometric progression exist? Find the sum to infinity.

[3 marks]

[**Total: 10 marks**]

5. The line ℓ is perpendicular to the line $y = x+5$ and passes through the point $(2,5)$. A curve has equation $y = (x-1)(x-3)$.

(a) Find the equation of ℓ.

[2 marks]

(b) Sketch the line ℓ and the curve on the same axes, marking clearly all intercepts and the coordinates of the turning point of the curve.

[3 marks]

(c) The line ℓ and the curve intersect at two points A and B.

(i) Find the coordinates of A and B.

Leaving your answer in the simplest surd form, find the distance:

(ii) between A and B;

(iii) of the y-intercept of the curve to the line ℓ.

[3, 1, 1 marks]

[**Total: 10 marks**]

6. (a) The points P, Q and R lie on a semicircle with radius 5 cm, and with P and Q diametrically opposite, as shown in Figure 1.

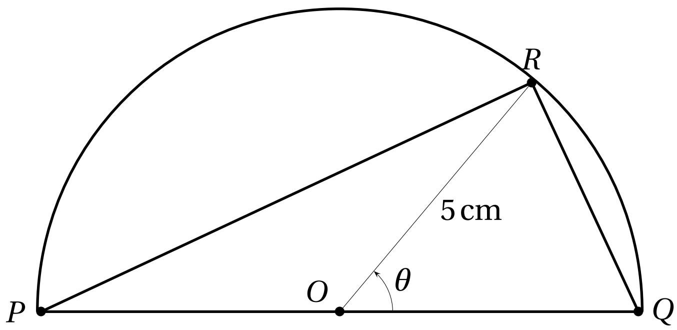

Figure 1

(i) Show that if θ is the angle subtended at the centre O by the chord RQ, the area of triangle PRQ is equal to $25\sin\theta\,\text{cm}^2$.

(ii) If the point R is allowed to move on the semicircle, what is maximum area that triangle PQR may have?

[3, 3 marks]

(b) Prove the identity $\operatorname{cosec}\theta\cot\theta + \sec\theta \equiv \operatorname{cosec}^2\theta\sec\theta$.

[3 marks]

(c) Solve for θ between $-\pi$ and π the equation

$$\frac{5}{3-\sin\theta} = 2.$$

[3 marks]

[**Total: 12 marks**]

7. (a) The letters of the word PHILADELPHIA are arranged in a row.

(i) In how many ways can this be done?

How many of these permutations have:

(ii) the P's next to each other?

(iii) the vowels next to each other?

[2, 2, 2 marks]

(b) The probability that Tom misses the school bus in the morning is 1/10. By using a tree diagram, or otherwise, find the probability that out of three consecutive days, Tom misses the morning-bus exactly once.

[4 marks]

[**Total: 10 marks**]

8. (a) Find the constants a, b, c, d and e given that the 3×3 matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} a & 1 & b \\ 1 & c & 1 \\ d & e & 4 \end{pmatrix}$$

are the inverse of each other.

[5 marks]

(b) Using the binomial expansion, expand $\left(1+\dfrac{x}{2}\right)^9$ in ascending powers of x up to and including the term in x^3. Using an appropriate value of x in this expansion, obtain an approximate estimate of 1.05^9.

[3, 2 marks]

[Total: 10 marks]

9. (a) Find the derivative of $\cos^2(3x)$.

[3 marks]

(b) Hence, or otherwise, solve the differential equation

$$y\cos^2(3x)\frac{\mathrm{d}y}{\mathrm{d}x} = (1-3y^2)\sin(3x)\cos(3x),$$

given that $y = \sqrt{3}$ when $x = 0$.

[7 marks]

[Total: 10 marks]

10. Let $f(x) = x^2 - 3x - 10$ and $g(x) = 5 - 4x - x^2$.

(a) On the same set of axes, sketch the curve $\mathscr{C}_1$ with equation $y = f(x)$ and the curve $\mathscr{C}_2$ with equation $y = g(x)$, showing clearly where the two curves cut the coordinate axes.

[4 marks]

(b) Find the x-coordinate of the two points of intersection of $\mathscr{C}_1$ and $\mathscr{C}_2$.

[2 marks]

(c) Evaluate the area enclosed between the two curves, giving your answer correct to 3 decimal places.

[5 marks]

[Total: 11 marks]